Oct. 8, 2009

Mike Hopkins "The $\mathbb{A}^1$ connectivity theorems"

Stable connectivity thm.
(connectivity thm for spectra)

$\mathcal{L}$ = category of spectra in

$s\, Sh(Sm)_{NIS}$ ← Justin purity thm. →

sm - smooth schemes over a field K

$E \in \mathcal{L}$ $\quad E = \{E_0, E_1, \dots$
$\{\Sigma E_n \to E_{n+1}\}\}$

$\Sigma X = S^1_0 \wedge X$
↑ standard simplicial S^1

$= \Delta[1]/\partial\Delta[1]$

weak equiv = stalk wise weak equiv

$\mathbb{A}^1$-local homotopy theory

Stable connectivity Thm

If $E \in \mathcal{L}$ is $[n-1]$ connected then so is $L^{\mathbb{A}^1} E$.

Purity Thm $\quad Z \subset X$ smooth

N = normal bundle

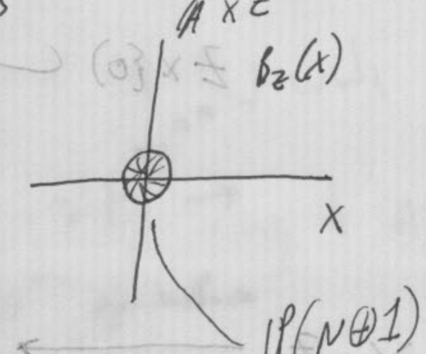
$X/X-Z \underset{\mathbb{A}}{\sim} \mathrm{Thom}(N)$
$\qquad\qquad\qquad \| $
$\qquad\qquad\quad N/N-\{0\}$

Sketch proof of purity

"Deformation to the normal cone" ← Google it!

Blow up $X \times \mathbb{A}^1$ at $Z \times \{0\}$

$B_Z(X)$



$B_Z(X)/(B_Z(X) - Z \times \mathbb{A}^1)$

$X/X-Z \times \{1\}$

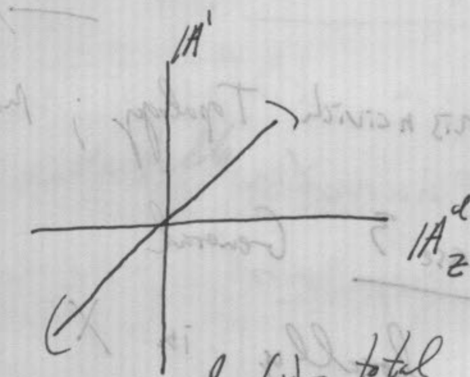
∪

$N = \mathbb{P}(N \oplus 1)$
$= \mathbb{P}(N)$

Claim These maps are $\mathbb{A}^1$ equivalences.

Case 1 $\quad Z \subset X$

$Z \subset Z \times \mathbb{A}^d$
↑ affine space



$B_Z(X)$

$\mathcal{L} \qquad \mathcal{L}_{x_0}$
↓
$\mathbb{P}^n \supset x_0$

$B_Z(X)$ = total space of the tautological line bundle over projective space.

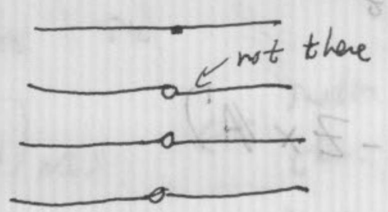
$B_Z(X) - Z \times \mathbb{A}^1 = \mathcal{L} - \mathcal{L}_{x_0}$

$\mathbb{A}^n/\mathbb{A}^n - \{0\}$
$N/N-\{0\}$
$\|$

In this case $\{\mathcal{L}/\mathcal{L} - \mathcal{L}_{x_0}\}$
↓ $\mathbb{A}^1$ - homotopy equivalence
$\{\mathbb{P}^n/\mathbb{P}^n \setminus x_0\} \leftarrow \mathbb{A}^n/\mathbb{A}^n - \{0\}$

Case 2

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, equal, "\text{pull back}"'] & X \arrow[d, "\text{étale}"] \\ Z \times \{0\} \arrow[r, hook] & Z \times \mathbb{A}^d \end{tikzcd}$$



$$\begin{tikzcd} X - Z \arrow[r] \arrow[d] & X \arrow[d, "\text{étale}"] \\ Z \times \mathbb{A}^d \arrow[r, "\text{open}"] & Z \times \mathbb{A}^d \end{tikzcd}$$

$Z \times \mathbb{A}^d \setminus Z \times \{0\}$

"distinguished square"

$$\Longrightarrow X / X - Z \simeq Z \times \mathbb{A}^d / Z \times (\mathbb{A}^d - \{0\})$$

Misnaevich. Topology ; Morell & Voivotsky

Case 3 General

locally in X we're in case (2)

Explicit construction of $L^{\mathbb{A}^1}$

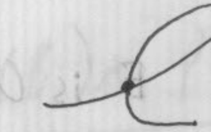
$S^0 = pt \xrightarrow{0} \mathbb{A}^1$ base point

$\mathbb{A}^1$

$\mathbb{A}^1$ with a disjoint basepoint

$\mathbb{A}^1_+$

$S^0 \xrightarrow{1} \mathbb{A}^1 \to C$ "Circle" with bad aim



$S^1 \wedge \sigma \sim C$

Internal hom $S^0 \to \underline{E}$

$$E \to E^\sigma \to ((E)^\sigma)^\sigma = E^{\sigma \wedge \sigma} \cdots \to E^{\sigma^{\wedge n}} \to$$

Proposition (1) $E \to E^\sigma$ is an $\mathbb{A}^1$ equivalence

(2) $\lim_{n\to\infty} E^{\sigma^{\wedge n}}$ is $\mathbb{A}^1$ local

$\lim_{n\to\infty} E^{\sigma \wedge n}$ is $L^{\mathbb{A}^1} E$

Pf (1) $E^{\mathbb{A}^1} \to E^{S^0} \to E^\sigma$

$E^{S^0} = E$

suffices to show $E^{\mathbb{A}^1}$ is $\mathbb{A}^1$ contractible

$$\mathbb{A}^1_+ \wedge E^{\mathbb{A}^1} \to E^{\mathbb{A}^1}$$

$$t, f \to f_t(s) = f(st)$$

$(\mathbb{A}^1 \times E)$ $\qquad \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$

Part (2) is LTR

$$\begin{tikzcd} U \times \{0\} \arrow[d] \\ U \times \mathbb{A}^1 \end{tikzcd}$$

$$[U_+ \wedge \mathbb{A}^1, \lim E^{\sigma^{\wedge n}}] = 0$$

$$\begin{tikzcd} U_+ \wedge \mathbb{A}^1 \arrow[r] \arrow[dr, "*"'] & E \arrow[d] \\ & E^\sigma \end{tikzcd}$$

"contract back"

Stable connectivity Thm

suffices to consider the case E is 0-connected.

To show $L^{\mathbb{A}^1} E$ is 0-connected

Like to prove: E 0-connected

$\Rightarrow E^{\tau}$ is 0-connected.

Lemma Suppose E is a spectrum s.t. $\forall$ smooth

U of dimension d.